11.8. PARABOLIC EQUATIONS

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x, t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions (Fig. 11.24).

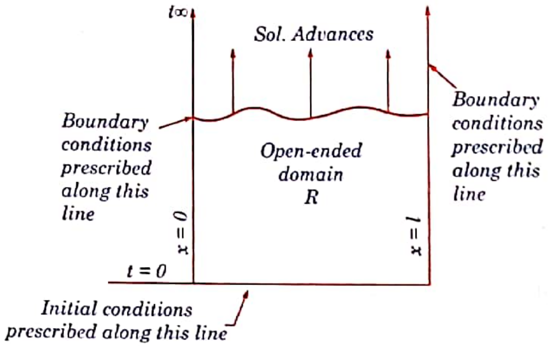

Fig. 11.24

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

11.9. SOLUTION OF ONE DIMENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

(1) Schmidt method. Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \text{[by (5) § 11.3}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad \text{[by (4) § 11.3}$$

Substituting these in (1), we obtain $u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$

or

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha u_{i+1,j} \qquad \text{...(2)}$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j + 1$ and j and is therefore, called a *2-level formula*. In schematic form (2) is shown in Fig. 11.25.

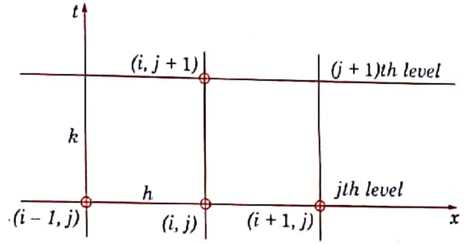

Fig. 11.25

Hence (2) is called the *Schmidt explicit formula* which is valid only for $0 < \alpha \le \frac{1}{2}$.

Obs. In particular when $\alpha = \frac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad \text{...(3)}$$

which shows that *the value of u at x_i at time t_{j+1} is the mean of the u-values at x_{i-1} and x_{i+1} at time t_j*. This relation, known as *Bendre-Schmidt recurrence relation*, gives the values of u at the internal mesh points with the help of boundary conditions.

(2) Crank-Nicolson method. We have seen that the Schmidt scheme is computationally simple and for convergent results $\alpha \le \frac{1}{2}$ *i.e.* $k \le h^2/2c^2$. To obtain more accurate results, h should be small *i.e.* k is necessarily very small. This makes the computations exceptionally lengthy as more time-levels would be required to cover the region. A method that does not restrict α and also reduces the volume of calculations was proposed by Crank and Nicolson in 1947.

According to this method, $\partial^2u/\partial x^2$ is replaced by the average of its central-difference approximations on the jth and $(j + 1)$th time rows. Thus (1) is reduced to

$$\frac{u_{i,j+1} - u_{i,j}}{h} = c^2 \cdot \frac{1}{2}\left\{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\} + \left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right\}$$

or

$$-\alpha u_{i-1,j+1} + (2 + 2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \qquad \text{...(4)}$$

where $\alpha = kc^2/h^2$.

Clearly the left side of (4) contains three unknown values of u at the $(j + 1)$th level while all the three values on the right are known values at the jth level. Thus (4) is a *2-level implicit relation* and is known as *Crank-Nicolson formula*. It is convergent for all finite values of α. Its computational model is given in Fig. 11.26.

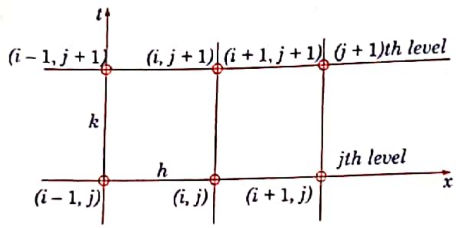

Fig. 11.26

If there are n internal mesh points on each row, then the relation (4) gives n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be solved to obtain the values at these mesh points. Similarly, the values at the internal mesh points on all rows can be found. A method such as this in which the calculation of an unknown mesh value necessitates the solution of a set of simultaneous equations, is known as an *implicit scheme*.

(3) Iterative methods of solution *for an implicit scheme.*

From (4), we have

$$(1 + \alpha)\, u_{i,j+1} = \tfrac{1}{2}\,\alpha(u_{i-1,j+1} + u_{i+1,j+1}) + u_{i,j} + \tfrac{1}{2}\,\alpha(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad ...(5)$$

Here only $u_{i,j+1}$, $u_{i-1,j+1}$ and $u_{i+1,j+1}$ are unknown while all others are known since these were already computed in the jth step.

Writing $\qquad b_i = u_{i,j} + \frac{\alpha}{2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$

and dropping j's (5) becomes $u_i = \frac{\alpha}{2(1+\alpha)}(u_{i-1} + u_{i+1}) + \frac{b_i}{1+\alpha}$

This gives the iteration formula

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\{u_{i-1}^{(n)} + u_{i+1}^{(n)}\} + \frac{b_i}{1+\alpha} \quad ...(6)$$

which expresses the $(n + 1)$th iterates in terms of the nth iterates only. This is known as the *Jacobi's iteration formula.*

As the latest value of u_{i-1} *i.e.* $u_{i-1}^{(n+1)}$ is already available, the convergence of the iteration formula (6) can be improved by replacing $u_{i-1}^{(n)}$ by $u_{i-1}^{(n+1)}$. Accordingly (6) may be written as

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\{u_{i-1}^{(n+1)} + u_{i+1}^{(n)}\} + \frac{b_i}{1+\alpha} \quad ...(7)$$

which is known as *Gauss-Seidal iteration formula.*

Obs. Gauss-Seidal iteration scheme is valid for all finite values of α and converges twice as fast as Jacobi's scheme.

(4) Du Fort and Frankel method. If we replace the derivatives in (1) by the central-difference approximations,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \text{[From (7) § 11.3}$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j-1} + u_{i+1,j}}{h^2} \qquad \text{[From (4) § 11.3}$$

we obtain $$u_{i,j+1} - u_{i,j-1} = \frac{2kc^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

i.e. $$u_{i,j+1} = u_{i,j-1} + 2\alpha\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] \quad ...(8)$$

where $\alpha = kc^2/h^2$. This difference equation is called the *Richardson scheme* which is a *3-level method.*

If we replace $u_{i,j}$ by the mean of the values $u_{i,j-1}$ and $u_{i,j+1}$ *i.e.* $u_{i,j} = \frac{1}{2}(u_{i,j-1} + u_{i,j+1})$ in (8), then we get

$$u_{i,j+1} = u_{i,j-1} + 2\alpha[u_{i-1,j} - (u_{i,j-1} + u_{i,j+1}) + u_{i+1,j}]$$

On simplification, it can be written as

$$u_{i,j+1} = \frac{1-2\alpha}{1+2\alpha}\,u_{i,j-1} + \frac{2\alpha}{1+2\alpha}[u_{i-1,j} + u_{i+1,j}] \quad ...(9)$$

This difference scheme is called *Du Fort-Frankel method* which is a *3-level explicit method.* Its computational model is given in Fig. 11.27.

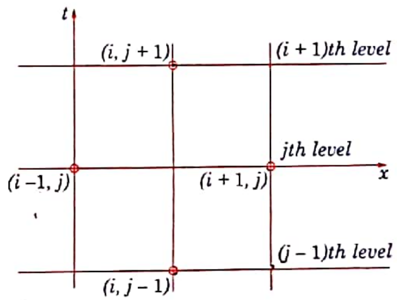

Fig. 11.27

■ **Example 11.9.** Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x^2}$ *in* $0 < x < 5$, $t \geq 0$ *given that* $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. *Compute u for the time-step with h = 1 by Crank-Nicholson method.*

(Anna, B. Tech., 2006)

Sol. Here $\quad c^2 = 1$ and $h = 1$.

Taking α (*i.e.* c^2k/h) = 1, we get $k = 1$.

Also we have

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$\therefore$ $4u_1 = 0 + 20 + 0 + u_2$ i.e. $4u_1 - u_2 = 20$ ...(1)

$4u_2 = 20 + 20 + u_1 + u_3$ i.e. $u_1 - 4u_2 + u_3 = -40$ ...(2)

$4u_3 = 20 + 20 + u_2 + u_4$ i.e. $u_2 - 4u_3 + u_4 = -40$ ...(3)

$4u_4 = 20 + 100 + u_3 + 100$ i.e. $u_3 - 4u_4 = -220$ ...(4)

Now (1) – 4(2) gives $15u_2 - 4u_3 = 180$ ...(5)

4(3) + (4) gives $4u_2 - 15u_3 = -380$ ...(6)

Then 15(5) – 4(6) gives $209\,u_2 = 4220$ i.e. $u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180$ i.e. $u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2$ i.e. $u_1 = 10.05$

From (4), $4u_4 = 220 + 30.75$ i.e. $u_4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.68.

Example 11.10. *Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ using Schmidt method (Take $h = 0.2$ and $\alpha = 1/2$).*
(V.T.U., B.E., 2013)

Sol. Since $h = 0.2$ and $\alpha = 1/2$

$\therefore$ $\alpha = \dfrac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(i_{i-1,j} + u_{i+1,j}) \quad ...(i)$$

We have $u(0, 0) = 0$, $u(0.2, 0) = \sin \pi/5 = 0.5875$

$u(0.4, 0) = \sin 2\pi/5 = 0.9511$, $u(0.6, 0) = \sin 3\pi/5 = 0.9511$

$u(0.8, 0) = \sin 4\pi/5 = 0.5875$, $u(1, 0) = \sin \pi = 0$

The values of u at the mesh points can be obtained by using the recurrence relation (i) as shown in table below :

$t \downarrow$ / $x \to$		0	0.2	0.4	0.6	0.8	1.0
	j \ i	0	1	2	3	4	5
0	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Example 11.11. *Find the values of $u(x, t)$ satisfying the parabolic equation $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i$: $i = 0, 1, 2, ..., 7$ and $t = \frac{1}{8}j : j = 0, 1, 2, ..., 5$.*

Sol. Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

$\therefore$ We have Bendre-Schmidt's recurrence relation $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$ ...(i)

Now since $u(0, t) = 0 = u(8, t)$

$\therefore$ $u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j, i.e. the entries in the first and last column are zero.

Since $u(x, 0) = 4x - \frac{1}{2}x^2$

$\therefore$ $u_{i,0} = 4i - \frac{1}{2}i^2$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ for $i = 0, 1, 2, 3, 4, 5, 6, 7$ at $t = 0$

These are the entries of the first row.

Putting $j = 0$ in (i), we have $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking $i = 1, 2, ..., 7$ successively, we get

$$u_{1,1} = \tfrac{1}{2}(u_{0,0} + u_{2,0}) = \tfrac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \tfrac{1}{2}(u_{1,0} + u_{3,0}) = \tfrac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \tfrac{1}{2}(u_{2,0} + u_{4,0}) = \tfrac{1}{3}(6 + 8) = 7$$

$$u_{4,1} = 7.5,\ u_{5,1} = 7,\ u_{6,1} = 5.5,\ u_{7,1} = 3.$$

These are the entries in the second row.

Putting $j = 1$ in (i), the entries of the third row are given by

$$u_{1,2} = \tfrac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (i), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as given in the following table :

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

■ **Example 11.12.** *Solve the equation* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ *subject to the conditions* $u(x, 0) = \sin \pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$, *using (a) Schmidt method, (b) Crank-Nicolson method, (c) Du Fort-Frankel method. Carryout computations for two levels, taking* $h = 1/3$, $k = 1/36$.

(V.T.U., B.E., 2014)

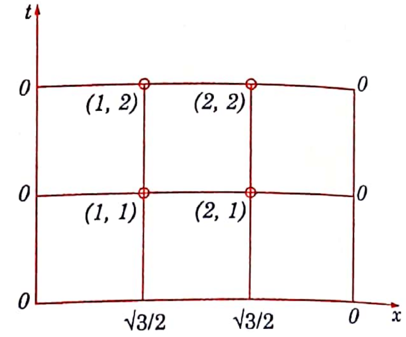

Fig. 11.28

Sol. Here $c^2 = 1$, $h = 1/3$, $k = 1/36$ so that $\alpha = kc^2/h^2 = 1/4$.

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2$, $u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$ and all boundary values are zero as shown in Fig. 11.28.

(*a*) *Schmidt's formula* [(2) of § 11.9]

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}$$

becomes $$u_{i,j+1} = \tfrac{1}{4}\,[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$$

For $i = 1, 2 ; j = 0$:

$$u_{1,1} = \tfrac{1}{4}\,[u_{0,0} + 2u_{1,0} + u_{2,0}] = \tfrac{1}{4}\,(0 + 2 \times \sqrt{3}/2 + 3/2) = 0.65$$

$$u_{2,1} = \tfrac{1}{4}\,[u_{1,0} + 2u_{2,0} + u_{3,0}] = \tfrac{1}{4}\,(\sqrt{3}/2 + 2 \times \sqrt{3}/2 + 0) = 0.65$$

For $i = 1, 2 ; j = 1$:

$$u_{1,2} = \tfrac{1}{4}\,(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \tfrac{1}{4}\,(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49$$

(*b*) *Crank-Nicolson formula* [(4) of § 11.9] becomes

$$-\tfrac{1}{4}\,u_{i-1,j+1} + \tfrac{5}{2}\,u_{i,j+1} - \tfrac{1}{4}\,u_{i+1,j+1} = \tfrac{1}{4}\,u_{i-1,j} + \tfrac{3}{2}\,u_{i,j} + \tfrac{1}{4}\,u_{i+1,j}$$

For $i = 1, 2 ; j = 0$:

$$-u_{0,1} + 10u_{1,1} - u_{2,1} = u_{0,0} + 6u_{1,0} + u_{2,0}$$

i.e. $$10\,u_{1,1} - u_{2,1} = 7\sqrt{3}/2$$

$$-u_{1,1} + 10u_{2,1} - u_{3,1} = u_{1,0} + 6u_{2,0} + u_{3,0}$$

i.e. $$-u_{1,1} + 10u_{2,1} = 7\sqrt{3}/2$$

Solving these equations, we find

$$u_{1,1} = u_{2,1} = 0.67$$

For $i = 1, 2 ; j = 1$:

$$-u_{0,2} + 10u_{1,2} - u_{2,2} = u_{0,1} + 6u_{1,1} + u_{2,1}$$

i.e. $$10u_{1,2} - u_{2,2} = 4.69$$

$$-u_{1,2} + 10u_{2,2} - u_{3,2} = u_{1,1} + 6u_{2,1} + u_{3,1}$$

i.e.
$$-u_{1,2} + 10u_{2,2} = 4.69$$

Solving these equations, we get $u_{1,2} = u_{2,2} = 0.52$.

(c) *Du Fort-Frankel formula* [(8) of § 11.9] becomes $u_{i,\,j+1} = \frac{1}{3}\,(u_{i,\,j-1} + u_{i-1,\,j} + u_{i+1,\,j})$

To start the calculations, we need $u_{1,\,1}$ and $u_{2,\,1}$.

We may take $u_{1,\,1} = u_{2,\,1} = 0.65$ from Schmidt method.

For $i = 1, 2\,;\, j = 1$:

$$u_{1,\,2} = \frac{1}{3}\,(u_{1,\,0} + u_{0,\,1} + u_{2,\,1}) = \frac{1}{3}\,(\sqrt{3}/2 + 0 + 0.65) = 0.5$$

$$u_{2,\,2} = \frac{1}{3}\,(u_{2,\,0} + u_{1,\,1} + u_{3,\,1}) = \frac{1}{3}\,(\sqrt{3}/2 + 0.65 + 0) = 0.5.$$